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Dual synchronization of chaos

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This paper treats the problem of simultaneously synchronizing two different pairs of chaotic oscillators with a single scalar signal. The condition for dual synchronization is obtained explicitly for chaotic oscillators represented by specific classes of piecewise-linear maps with conditional linear coupling. Dual synchronization with conditional linear coupling is also demonstrated numerically for oscillators modeled by a number of different classes of maps, and for oscillators modelled by delay-differential equations.

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Chaos synchronization, or synchronization of chaotic oscillators, provides a means to copy chaos; that is, to generate identical chaotic oscillations in different sites, by coupling the oscillators with suitable driving signals [1,2]. The topic of synchronization of chaotic oscillators has attracted increased attention in recent years because of possible relevance to communications and biological systems [3,4]. One of the interesting developments concerns the possibility of synchronizing multiple pairs of oscillators using just one communication channel [5]. This is potentially useful in particular to applications of chaos to spectrum-spreading communication systems [6].

This work concentrates on using a scalar signal to simultaneously synchronize two different pairs of chaotic oscillators, which we refer to as dual synchronization. Figure 1 is a schematic circuit diagram showing the situation of dual synchronization. The outputs of a pair of master oscillators are linearly coupled and fed to a pair of slave oscillators. The signals from the slave oscillators are coupled in a similar way and subtracted from the signal received from the masters' and the difference signal, or the joint error signal, is injected into each slave oscillator. When the slaves are synchronized to their respective masters, the joint error signal is zero and no signal is injected into the slaves, so they are free oscillating. The fact that there is no coupling between the two master oscillators distinguishes this problem from the problem of using a single scalar signal to synchronize multidimensional chaotic oscillators, or hyperchaotic oscillators with multiple positive Lyapunov exponents [7]. Tsimring and Sushchik [5] showed that dual synchronization is possible for oscillators modeled by some well-known discrete maps when the contributions to the common signal are equal, i.e., $\varepsilon_1=\varepsilon_2=1/2$ in Fig. 1. An explicit analytic condition for synchronization was obtained for maps known as tent maps.

In this Rapid Communication, we show further, the proof of dual chaos synchronization can be extended to the case of maps with coupling coefficients satisfying the linear condition $\varepsilon_1+\varepsilon_2=1$. The extension of the coupling condition facilitates synchronization between very different pairs of chaotic oscillators. We show numerically examples of dual synchronization over a wide range of parameters in the case of various different pairs of chaotic maps, including the logistic map, Chebyshev map, generalized tent map, and a class of cosine maps. The extension of the coupling condition also facilitates the implementation of dual chaos synchronization in practical physical systems. We propose a scheme of performing dual chaos synchronization in two pairs of nonlinear resonators which can be modeled by delay-differential equations. The robustness of dual chaos synchronization in delay-differential systems with respect to both parameter mismatches and additive noises is verified.

To start with, we consider the case of a pair of masters X and Y sending signals to a pair of slaves x and y using a common channel in which their signals are linearly coupled.

$$X(t+1)=f(X(t)), \tag{1}$$

$$Y(t+1)=g(Y(t)). \tag{2}$$

Here, we consider the coupling in a general way by linearly combining the two outputs of the master oscillators as

$$u(t)=\varepsilon_1 f(X(t))+\varepsilon_2 g(Y(t)), \tag{3}$$

where $\varepsilon_1,\ \varepsilon_2 (0\leqslant\varepsilon_1,\ \varepsilon_2\leqslant 1)$ are coupling parameters. The slave system contains two oscillators identical to the pair on the master side and each oscillator is injected with an error signal $e(t)$,

$$x(t+1)=f(x(t))+e(t), \tag{4}$$

$$y(t+1)=g(y(t))+e(t), \tag{5}$$

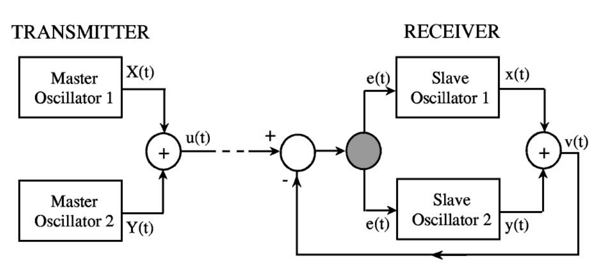

FIG. 1. Schematic diagram of dual synchronization. Signals from two independent master oscillators, represented by X and Y, are sent to a system containing two corresponding slave oscillators, represented by x and y. In the dual synchronization state, $x=X$, $y=Y$.

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with

$$e(t)=u(t)-v(t), \tag{6}$$

$$v(t)=\varepsilon_1 f(x(t))+\varepsilon_2 g(y(t)). \tag{7}$$

The dual synchronization state is defined as $x(t)=X(t)$, $y(t)=Y(t)$. Clearly such a dual synchronization state can exist as a solution. For example, if the initial state is chosen so $x(0)=X(0)$ and $y(0)=Y(0)$, the error signal is zero and remains zero, so the oscillations are and remain identical. We next show that the dual synchronization state can also be an attracting solution by evaluating the Lyapunov exponent of the slave system with respect to the synchronized state $x(t)=X(t)$, $y(t)=Y(t)$.

Assume a small perturbation at time t is $\delta x(t)=x(t)-X(t)$ and $\delta y(t)=y(t)-Y(t)$. Such perturbation evolves according to the linearized dynamics given by

$$[\delta x(t+1),\delta y(t+1)]^T=\mathbf{M}(t)[\delta x(t),\delta y(t)]^T \tag{8}$$

where T means transpose and

$$\mathbf{M}(t)=\begin{bmatrix}(1-\varepsilon_1)D_f(t) & -\varepsilon_2 D_g(t)\\ -\varepsilon_1 D_f(t) & (1-\varepsilon_2)D_g(t)\end{bmatrix} \tag{9}$$

is a 2×2 Jacobian matrix with $D_f(t)\equiv df/dx|_{x=X(t)}$ and $D_g(t)\equiv dg/dy|_{y=Y(t)}$.

In Ref. [5], dual synchronization was analytically proven for a special coupling case, $\varepsilon_1=\varepsilon_2=1/2$. Here, we show that such coupling constraint could be extended to a line $\varepsilon_1+\varepsilon_2=1$. Under this condition, one of the eigenvalues of $\mathbf{M}$ is identically zero, and the only nonzero eigenvalue is given simply by

$$\gamma=(1-\varepsilon_1)D_f+\varepsilon_1 D_g. \tag{10}$$

The corresponding eigenvector (Λ_x,Λ_y) satisfies $\varepsilon_1\Lambda_x+(1-\varepsilon_1)\Lambda_y=0$ and depends only on the ratio of the two coupling coefficients, remaining constant during the evolution of the slave system. The maximum Lyapunov exponent λ is then given by $\lambda=\lim_{L\to\infty}1/L\Sigma_{t=0}^{L-1}\ln|\gamma(t)|$. Thus, we obtain the condition for dual synchronization with the linear coupling as

$$\lim_{L\to\infty}\frac{1}{L}\sum_{t=0}^{L-1}\ln|(1-\varepsilon_1)D_f(t)+\varepsilon_1 D_g(t)|<0. \tag{11}$$

Now when the oscillation of the master oscillators is ergodic, we can replace the average over time by an average over the variables X and Y using the invariant density $\rho(X,Y)$ to express the condition as

$$\int\int\ln|(1-\varepsilon_1)D_f(X)+\varepsilon_1 D_g(Y)|\rho_f(X)\rho_g(Y)dXdY<0. \tag{12}$$

Here we use the fact that the dynamics of the two masters are not correlated, so $\rho(X,Y)=\rho_f(X)\rho_g(Y)$, where $\rho_f(X)$ and $\rho_g(Y)$ are the invariant densities of the two master oscillators. Equation (12) gives the general condition for dual synchronization of two pairs of one-dimensional maps with linear coupling $\varepsilon_1+\varepsilon_2=1$.

TABLE I. Chaos maps used in numerical experiments of dual synchronization.

Map	$f(x)$	$\rho(x)$
Tent	$(-1)^{[qx]}qx$ mod1	1 $(q=2,3\ldots)$
Chebyshev	$\cos(q\cos^{-1}x)$	$1/\pi\sqrt{1-x^2}$ $(q=2,3\ldots)$
Logistic	$qx(1-x)$	$1/\pi\sqrt{x(1-x)}$ $(q=4)$
Cosine	$\mu\cos(x+\theta)$	numerically available

For dual synchronization of two pairs of chaotic oscillators, we need to satisfy Eq. (12) even though each master is independently chaotic with $\int\ln|D_f(X)|\rho_f(X)dX>0$ and $\int\ln|D_g(Y)|\rho_g(Y)dY>0$. Note that even if $|D_f|$ and $|D_g|$ are both greater than unity in magnitude, when D_f and D_g have opposite signs, $|(1-\varepsilon_1)D_f+\varepsilon_1 D_g|$ may be smaller than unity, so the coupling of the slave oscillators can reduce the magnitude of the deviation, and thus facilitate dual synchronization. Certainly it can be seen that dual synchronization, for example, is not possible for maps in which D_f and D_g are both greater than unity everywhere.

We give some specific examples where the condition for dual synchronization can be analytically obtained for the conditional coupling $\varepsilon_1+\varepsilon_2=1$. The first one is two pairs of identical oscillators represented by generalized tent maps, i.e., $f(x)=g(x)=(-1)^{[qx]}qx$ mod1, with $q=2,3,4,\ldots$, where $[qx]$ is the integer part of qx. Since the invariant density of the tent map is unity over the domain [8], one can easily verify that $|\gamma_1(t)|=|(1-\varepsilon_1)D_f(t)+\varepsilon_1 D_g(t)|$ only has two possible values as $|1-2\varepsilon_1|q$ and q with the probability of $[q^2/2]/q^2$ and $[(q^2+1)/2]/q^2$, respectively. Then the maximum Lyapunov exponent is given by $\lambda=(1/\chi)\ln(q^\chi|1-2\varepsilon_1|)$, where $\chi=q^2/[q^2/2]$. This yields the condition for dual synchronization of two pairs of tent maps as $(1-q^{-\chi})/2<\varepsilon_1<(1+q^{-\chi})/2$. For the usual tent map at $q=2$, the condition is $3/8<\varepsilon_1<5/8$.

Let us next consider the Bernoulli shift map $x(t+1)=2x(t)$mod1, which also has a uniform invariant density. In the case of two pairs of Bernoulli shift maps, $f(x)=g(x)=2x$ mod1, γ_1 is always 2 and dual synchronization can never be achieved. However, if one chooses $f(x)=2x$ mod1 and $g(x)=(-1)^{[qx]}qx$ mod1 with q restricted to be an even number, γ_1 has two possible values as $2(1-\varepsilon_1)+q\varepsilon_1$ and $2(1-\varepsilon_1)-q\varepsilon_1$ with equal probability and one then obtains $\lambda=1/2\ln|4(1-\varepsilon_1)^2-q^2\varepsilon_1^2|$. The condition for dual synchronization is $3/8<\varepsilon_1<5/8$ for $q=2$ and $(\sqrt{3q^2+4}-4)/(q^2-4)<\varepsilon_1<(\sqrt{5q^2-4}-4)/(q^2-4)$ for $q>2$, with the strongest synchronization ($\lambda=-\infty$) at $\varepsilon_1=2/(q+2)$ for both cases. It is worth noting that for $q>2$, dual chaos synchronization is not possible at $\varepsilon_1=\varepsilon_2=1/2$.

We have done numerical tests of dual synchronization using a number of different maps, including the logistic map, Chebyshev map, generalized tent map, and cosine map. These maps together with their available invariant density [8] are listed in Table I. For logistic and Chebyshev maps, the perturbations due to coupling may take the map out of its usual domain, so we extended the domain by making the

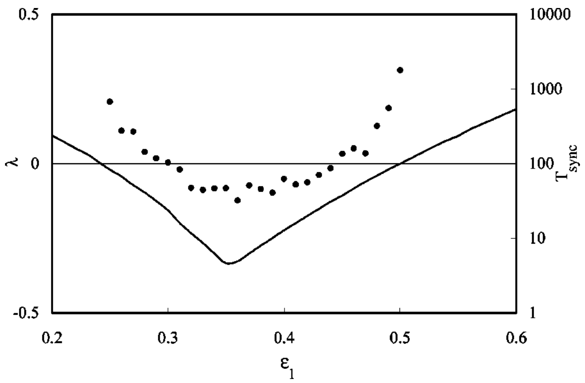

FIG. 2. Lyapunov exponent λ and synchronization time T_{sync} as functions of coupling coefficient for conditional coupling $\varepsilon_1+\varepsilon_2=1$. Solid line and circles denote λ and T_{sync} respectively for dual synchronization of two different pairs of chaotic oscillators: a pair of cosine maps ($\mu=2.2$) and a pair of logistic maps ($q=4$). Parameter range for successful dual synchronization corresponds to the range for negative Lyapunov exponent.

map periodic, i.e., we take $f(x)=f(x\pm n)$ for the logistic map and take $f(x)=f(x\pm 2n)$ for the Chebyshev map, where n is an integer. It was verified that for almost all pairs in Table I, there exists a parameter range over which the Lyapunov exponent at the dual synchronized state is negative. The only exceptional case is the coupling between two pairs of Chebyshev maps where λ never becomes negative, implying no dual synchronism happens in this case.

Figure 2 shows both the Lyapunov exponent λ and the synchronization time T_{sync} as functions of the coupling coefficient for dual synchronization of two pairs of chaotic oscillations generated from two different maps: a pair of cosine maps together with a pair of logistic maps $[f(x)=\mu\cos(x)$ and $g(x)=qx(1-x)]$. Here, T_{sync} is defined to be the average time for the error signal between the slave and master systems, $\mathrm{Err}(t)=|x(t)-X(t)|+|y(t)-Y(t)|$, to become less than a certain magnitude $\delta(\equiv 10^{-6}$ in Fig. 2). As can be seen from the figure, there exists a wide range of the coupling coefficients over which λ is negative and dual synchronization succeeds. It was further verified that $T_{sync}\propto 1/|\lambda|$. The results demonstrate that the possibility of dual synchronization of two pairs of chaos maps is rightly guided by the condition Eq. (12) on the Lyapunov exponent calculated over separate chaotic attractors. We also note the fastest dual synchronization happens at $\varepsilon_1=0.35$, $\varepsilon_2=0.65$ rather than at $\varepsilon_1=\varepsilon_2=1/2$. It can be generally concluded that $\varepsilon_1=\varepsilon_2=1/2$ is not necessarily the optimal coupling for dual synchronizing of two different chaotic attractors.

In the second part of this paper, we discuss dual synchronization in chaotic systems described by a class of delay-differential equations of one variable, for which the mechanism for dual synchronization is related to that of one-dimensional maps. We consider the master system is described by two delay-differential equations with different nonlinearities as,

$$\tau dX(t)/dt+X(t)=f(X(t-T_r)), \tag{13}$$

$$\tau dY(t)/dt+Y(t)=g(Y(t-T_r)), \tag{14}$$

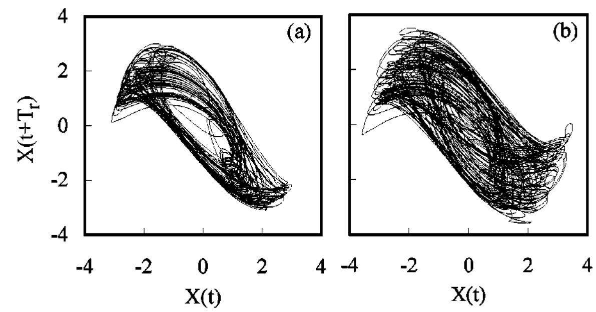

FIG. 3. Attractors of (a) chaos oscillator 1 and (b) chaos oscillator 2 used in dual synchronization. $\mu_1=3.0$, $\theta_1=0.4\pi$, $\mu_2=3.5$, $\theta_2=0.5\pi$, $T_r/\tau=100$.

where f and g are nonlinear functions, τ and T_r are respectively the response time and the time delay in the feedback.

The synchronization signal is generated by coupling outputs from the two master oscillators as $u(t)=\varepsilon_1 f(X(t))+\varepsilon_2 g(Y(t))$. Meanwhile, the slave system possesses the same set of oscillators with similar parameter values as those in the driver side, i.e.,

$$\tau dx(t)/dt+x(t)=f(x(t-T_r))+e(t-T_r), \tag{15}$$

$$\tau dy(t)/dt+y(t)=g(y(t-T_r))+e(t-T_r). \tag{16}$$

where $e(t)=u(t)-v(t)$ and $v(t)=\varepsilon_1 f[x(t)]+\varepsilon_2 g[y(t)]$. Here, the delay time is assumed to be identical for all oscillators.

Let us consider the dynamics of small perturbations about the dual synchronization state. Note that under the condition $\epsilon_1=\epsilon_2=0.5$ we can write

$$\tau d(\delta x+\delta y)/dt=-(\delta x+\delta y), \tag{17}$$

showing that there is convergence to the line $\delta x=-\delta y$. Then the condition for dual synchronization depends only on the convergence to zero of $\delta x(=-\delta y)$ on this line, which is governed by

$$\tau d\,\delta x(t)/dt=-\delta x(t)+0.5[D_f(t-T_r)+D_g(t-T_r)]\,\delta x(t-T_r). \tag{18}$$

Here $D_f(t)\equiv df/dX$ and $D_g(t)\equiv dg/dY$. These are time dependent but $\delta x(t)$ will tend to relax to zero if the second term on the right hand side is zero on average. This equation shows that in the case of this type of delay-differential oscillators, as in the case of the one-dimensional discrete maps, the possibility of stable dual synchronization of chaos is governed by the statistical balance of the fluctuating values of D_f and D_g.

Now we show numerically that there are particular delay-differential systems for which dual synchronization is possible. We consider delay-differential equations describing a well-known class of nonlinear resonator with a delayed feedback [9]. A typical form for f (and g) corresponding to experimental systems [9] is the cosine map, $f(X;\mu,\theta)=\mu\cos(X+\theta)$, where θ is an offset parameter and μ is a parameter usually proportional to the external input power. Figure 3 shows an example of two chaotic attractors, obtained for two nonlinear resonators with different parameter

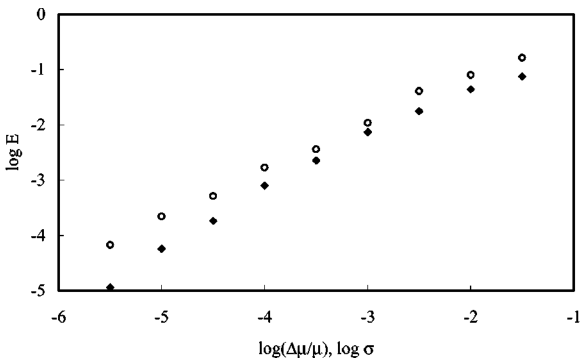

FIG. 4. Normalized synchronization error E vs parameter mismatch $\Delta\mu/\mu$ (open circles) and noise level σ (closed triangles) for $\mu_1=3.0$, $\theta_1=0.4\pi$, $\mu_2=3.5$, $\theta_2=0.5\pi$, $T_r/\tau=100$, $\varepsilon_1=0.4$, and $\varepsilon_2=0.6$. The base of the logarithm is 10.

values, for which dual synchronization is possible. It is found that two oscillators in the slave side synchronize to their corresponding oscillators in the master side within a time interval typically about $100T_r$. Numerical results show that dual synchronization is achieved over a wide range of parameters (μ,θ) and linear coupling coefficients $(\varepsilon_1,\varepsilon_2)$.

To evaluate the robustness of dual synchronization, we use a normalized synchronization error E which is defined as the ratio of the root-mean-square (rms) value of the synchronization error to the rms value of the chaotic waveform of the driver system, i.e., $E=[\sigma(x-X)+\sigma(y-Y)]/[\sigma(X)+\sigma(Y)]$. Figure 4 shows the normalized synchronization error E as a function of parameter mismatch $\Delta\mu/\mu$ and the noise level σ for dual synchronizing the two different chaos attractors shown in Fig. 3. It is demonstrated that the error increases almost linearly with both the parameter mismatch and the noise level. One percent of parameter mismatch results in the synchronization error of 8% while one percent of noise results in the error of about 4%. The results imply that the proposed dual synchronization in delay-differential systems is robust to both the parameter mismatches and the system noise, which is important for physical realization of synchronizing systems.

In conclusion, we have shown that dual synchronization is possible between two pairs of independent chaotic oscillators with a generalized coupling. For dual synchronization in discrete maps, we have shown analytically that dual synchronization is possible for a more general coupling than the condition described in the previous work [5]. Numerical simulations using various chaos maps verified the effectiveness of our analysis. It was shown that a particular class of practical physical systems described by delay-differential equations, nonlinear resonators which have been investigated in a large number of experiments on opto-electronic oscillators [9], can be dually synchronized. The effects of parameter mismatches and noise, which need to be dealt with in actual experiments, are evaluated and the results verified the robustness of the dual synchronization in such systems.

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